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Third-Order Equation for Two-Component Spinors

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Abstract

We present the properties of a two-component spinor field that obeys a third-order equation. It is separated into a massive part that corresponds closely to a Dirac field, and a massless part that obeys the Weyl equation. We discuss the interaction of such a field with an external electromagnetic field and the (weak) interactions of two such fields. They can be considered both in terms of relativistic quantum mechanics and quantum field theory. We conclude that this formulation has some attractive features, such as a unified treatment of electrons and muons with their neutrinos, a special role of the $\mathscr{P}\mathscr{C}$ transformation, a more convergent propagator and a new approach to interactions. It also has some serious difficulties, aside from those generally associated with higher-order equations. These are mainly related to inconsistencies in the simultaneous considerations of electromagnetic and weak interactions. The approach also suggests a further unification of the electron and muon fields into a single bispinor field.

1. *Introduction*

The new insight into the nature of weak interactions be Lee & Yang (1957) led to changes in the theory of the four-fermion interaction (Sudarshan & Marshak, 1958; Feynman & Gell-Mann, 1958). In particular, Feynman advocated the use of two-component spinors not only for the massless neutrino, but also for massive charged particles such as the electron. This generated a considerable amount of interest in the theory of these spinors (Brown, 1958; Kibble & Polkinghorne, 1958; G. Marx, 1958; Tonin, 1954; Theis, 1959; Barut & Mullen, 1962a), but the results were inconclusive and the usual formulation of the Dirac equation in terms of bispinors retained its preferential place in the literature.

One difficulty associated with the theory of two-component spinors is the lack of a Lagrangian density‡ that leads to a second-order wave equation. Attempts along these lines have led to the third-order equation

‡ A possible formulation for a quantized field is discussed by Case (1957).

(Kibble & Polkinghorne, 1958; Barut & Mullen 1962a) we discuss here. Another possibility that was explored (Marx, 1970a) was an observer-dependent Lagrangian density which allows a quantization by means of commutators, but does not describe the usual electrons.

We have two main reasons to come back to the third-order equation. One is the experimental verification of the existence of two kinds of neutrinos,‡ which makes the association of a massless particle to each charged lepton more attractive. The other is an increase in flexibility brought to the theory by developments in relativistic quantum mechanics (Marx, 1969, 1970a, b, c), since the quantization of the field presents difficulties such as those related to indefinite metrics in the space of state vectors.

In Section 2 we recast some results obtained from the Dirac equation in terms of two-component spinors. They are equivalent to a considerable extent, but some differences are apparent in relation to improper and antichronous Lorentz transformations presented in Section 3. We then derive the third-order equation via the Lagrangian density and find the corresponding conserved quantities in Section 4. For the free field, they can be separated into parts coming from the massive and massless fields. Section 5 contains a discussion of the interaction with a given electromagnetic field, introduced by means of the usual gauge-invariant substitution. This approach for a single field implies that the massless field also interacts with the electromagnetic one, which would present difficulties in its identification with a neutrino. Weak interactions are introduced through vector currents formed from an electron field and a muon field, which have the corresponding neutrinos already incorporated into them; we do this in Section 6, and point out some differences with the usual four-fermion coupling due to additional terms in these currents. In Section 7 we discuss in general terms the interpretation of these classical fields in relativistic quantum mechanics, and in Section 8 we do so for a theory of quantized fields. We conclude in Section 9 with a summary of the results and some indications for further research.

We use real four-vectors and the time-favouring metric in Minkowski space. Our units are such that

$$\hbar = 1, \qquad c = 1, \tag{1.1}$$

four-vectors and tensors have Greek indices that range from 0 to 3, and we use the modified summation convention for repeated lower Greek indices, that is,

$$a_\mu b_\mu \equiv a_0 b_0 - \mathbf{a}\cdot\mathbf{b} \tag{1.2}$$

Spinors carry capital Roman indices, which are dotted when the spinor transforms with the complex conjugate of the unimodular transformation that corresponds to a given Loretnz transformation (Corson, 1953; Rzewuski, 1964; Aharoni, 1965). There are some discrepancies in the definitions of the raising and lowering of indices; our choice is explained

‡ See Frazer (1966), where references to the original literature are given.

in Marx (1970a). The quantities $\sigma_{\mu}^{\dot{A}B}$ are invariant under combined transformation for the vector and spinor indices, and they are represented by the unit 2×2 matrix and the three Pauli matrices. A bispinor can be related to two spinors through the relation

$$\psi(x) = \begin{pmatrix} \varphi_{A}(x) \\ \chi^{\dot{A}}(x) \end{pmatrix} \tag{1.3}$$

where the indices specify the unimodular transformation that corresponds to a Lorentz transformation. The Dirac matrices are given by

$$\gamma_{\mu} = \begin{pmatrix} 0 & -\sigma^{*}_{\mu A\dot{B}} \\ -\sigma_{\mu}^{\dot{A}B} & 0 \end{pmatrix} \tag{1.4}$$

that is,

$$\gamma_{0} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{1.5}$$

We note that

$$(\sigma_{\mu \dot{A} B})^{*} = \sigma^{*}_{\mu A\dot{B}} = \sigma_{\mu \dot{B} A} \tag{1.6}$$

and we associate a second-rank spinor

$$a^{\dot{A}B} = a_{\mu}\,\sigma_{\mu}^{\dot{A}B} \tag{1.7}$$

to the four-vector a_{μ}. Furthermore, we simplify considerably the notation by suppressing the spinor indices where they can be restored without difficulty; this does *not* result in the use of a matrix notation, since the indices should be restored in their natural positions on the different quantities and alternate as lower and upper indices, For instance,

$$\varphi^{*}_{,\alpha}\,\sigma_{\alpha}\,\sigma_{\mu}\,\sigma_{\beta}\,\varphi_{,\beta} \equiv \varphi^{*}_{\dot{A},\alpha}\,\sigma_{\alpha}^{\dot{A}B}\,\sigma_{\mu\dot{C}B}\,\sigma_{\beta}^{\dot{C}D}\,\varphi_{D,\beta} \tag{1.8}$$

Since

$$\chi^{A}\,\varphi_{A} = -\,\chi_{A}\,\varphi^{A} \tag{1.9}$$

the position of the index on φ is not important as long as we have even number of indices, but we have to know whether it is an undotted or dotted index. A frequently used identity is

$$\sigma_{\mu}\,\sigma_{\nu} + \sigma_{\nu}\,\sigma_{\mu} = 2g_{\mu\nu} \tag{1.10}$$

which stands for

$$\sigma_{\mu\dot{A}B}\,\sigma_{\nu}^{\dot{C}B} + \sigma_{\nu\dot{A}B}\,\sigma_{\mu}^{\dot{C}B} = 2g_{\mu\nu}\,\delta_{\dot{A}}^{\dot{C}} \tag{1.11}$$

or

$$\sigma_{\mu\dot{A}B}\,\sigma_{\nu}^{\dot{A}C} + \sigma_{\nu\dot{A}B}\,\sigma_{\mu}^{\dot{A}C} = 2g_{\mu\nu}\,\delta_{B}^{C} \tag{1.12}$$

depending on the context. Other definitions and identities are given throughout the text.

2. *The Dirac Field*

Using the relationships (1.3) and (1.4), the Dirac equation

$$(-i\gamma\cdot\partial + m)\,\psi(x) = 0 \tag{2.1}$$

becomes the system of coupled equations

$$i\partial^{\dot{A}B}\varphi_B(x) = -m\chi^{\dot{A}}(x) \tag{2.2}$$

$$i\partial_{\dot{A}B}\chi^{\dot{A}}(x) = -m\varphi_B(x) \tag{2.3}$$

One possible approach to the use of two-component spinors retains both the fields φ and χ (Brown, 1958; Tonin, 1959); in this manner we are only dealing with a change in notation. Alternatively we can eliminate one of the fields and use only the remaining one. If we substitute χ from equation (2.2) into (2.3), we obtain the Klein–Gordon equation

$$(\partial_{\dot{A}B}\,\partial^{\dot{A}C} + m^2\,\delta_B^C)\,\varphi_C(x) = 0 \tag{2.4}$$

This second-order equation is equivalent to the two first-order equations (2.2) and (2.3) in the sense that there is a one-to-one correspondence between the solutions. We can use equation (1.12) to rewrite this equation in the form

$$(\partial^2 + m^2)\,\varphi_C(x) = 0 \tag{2.5}$$

Although the equations are the same, they lead to different interactions with the electromagnetic field by means of the gauge-invariant substitution‡

$$\partial_\mu \to D_\mu = \partial_\mu - ie\,A_\mu. \tag{2.6}$$

We can further substitute these fields in the conserved densities for the bispinor field. Thus, the current density

$$j_\mu^{(1)} = \bar{\psi}\gamma_\mu\psi \tag{2.7}$$

becomes

$$j_\mu^{(1)} = \varphi^*\sigma_\mu\varphi + \varphi^*_{,\alpha}\sigma_\alpha\sigma_\mu\sigma_\beta\,\varphi_{,\beta}/m^2 \tag{2.8}$$

the stress-energy tensor

$$T_{\mu\nu}^{(1)} = \tfrac{1}{2}i(\bar{\psi}\,\gamma_\mu\,\psi_{,\nu} - \bar{\psi}_{,\nu}\,\gamma_\mu\,\psi) \tag{2.9}$$

gives

$$T_{\mu\nu}^{(1)} = \tfrac{1}{2}i(\varphi^*\,\sigma_\mu\,\varphi_{,\nu} - \varphi^*_{,\nu}\,\sigma_\mu\,\varphi + \varphi^*_{,\alpha}\,\sigma_\alpha\,\sigma_\mu\,\sigma_\beta\,\varphi_{,\beta\nu}/m^2 - \varphi^*_{,\alpha\nu}\,\sigma_\alpha\,\sigma_\mu\,\sigma_\beta\,\varphi_{,\beta}/m^2) \tag{2.10}$$

and the angular momentum density

$$M_{\mu\nu\rho}^{(1)} = x_\nu\,T_{\mu\rho}^{(1)} - x_\rho\,T_{\mu\nu}^{(1)} + \tfrac{1}{4}\bar{\psi}(\gamma_\mu\,\sigma_{\nu\rho} + \sigma_{\nu\rho}\,\gamma_\mu)\,\psi \tag{2.11}$$

where

$$\sigma_{\mu\nu} = \tfrac{1}{2}i(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu) \tag{2.12}$$

is transformed into

$$M_{\mu\nu\rho}^{(1)} = x_\nu\,T_{\mu\rho}^{(1)} - x_\rho\,T_{\mu\nu}^{(1)} + \tfrac{1}{4}i[\varphi^*(\sigma_\mu\,\sigma_\nu\,\sigma_\rho - \sigma_\rho\,\sigma_\nu\,\sigma_\mu)\,\varphi + \varphi^*_{,\alpha}\,\sigma_\alpha(\sigma_\mu\,\sigma_\nu\,\sigma_\rho - \sigma_\rho\,\sigma_\nu\,\sigma_\mu)\,\sigma_\beta\,\varphi_{,\beta}/m^2] \tag{2.13}$$

The expression (2.10) for $T_{\mu\nu}$ contains second-order time derivatives;

‡ This distinction is involved in a comment (G. Marx, 1958) that the Feynman equation seems to violate the 'principle of minimum electromagnetic interaction'.

they can be eliminated by using the equation of motion to show that

$$\varphi_{,\alpha\beta} = (\partial_\alpha \partial_\beta - g_{\alpha\beta}\,\partial^2)\,\varphi - m^2 g_{\alpha\beta}\,\varphi \tag{2.14}$$

(It has an expression free of $\partial_0{}^2$ on the right-hand side.)

Alternatively, we can eliminate φ to obtain the equation of motion

$$(\partial^{\dot{B}A}\,\partial_{\dot{C}A} + m^2\,\delta^{\dot{B}}_{\dot{C}})\,\chi^{\dot{C}}(x) = 0 \tag{2.15}$$

and the conserved densities have the same form as those in equations (2.8), (2.10) and (2.13); they differ only by the position and type of the spinor indices.

3. *Lorentz Transformations*

Under a Lorentz transformation, vectors transform according to

$$x'_\mu = a_\mu{}^\nu\, x_\nu \tag{3.1}$$

where $(a_\mu{}^\nu)$ is a real pseudo-orthogonal matrix that satisfies

$$a_\mu{}^\lambda\, a_\nu{}^\rho\, g^{\mu\nu} = g^{\lambda\rho} \tag{3.2}$$

while spinors undergo a linear transformation

$$\varphi'_{\mathrm{A}} = s_{\mathrm{A}}{}^{\mathrm{B}}\,\varphi_{\mathrm{B}}, \tag{3.3}$$

where $(s_{\mathrm{A}}{}^{\mathrm{B}})$ is a complex unimodular matrix; it satisfies

$$\det(s_{\mathrm{A}}{}^{\mathrm{B}}) = 1 \tag{3.4}$$

The relationship between the two types of transformations is given by the invariance of the σ_μ,

$$a_\mu{}^\nu\, s^{*\dot{\mathrm{C}}}_{\dot{\mathrm{A}}}\, s_{\mathrm{B}}{}^{\mathrm{D}}\, \sigma_{\nu\dot{\mathrm{C}}\mathrm{D}} = \sigma_{\mu\dot{\mathrm{A}}\mathrm{B}} \tag{3.5}$$

and we solve for $a_\mu{}^\nu$ to obtain

$$a_\mu{}^\nu = \tfrac{1}{2} s^{*\dot{\mathrm{C}}}_{\dot{\mathrm{A}}}\, s_{\mathrm{B}}{}^{\mathrm{D}}\, \sigma^{\dot{\mathrm{A}}\mathrm{B}}_{\mu}\, \sigma^{\nu}_{\dot{\mathrm{C}}\mathrm{D}} \tag{3.6}$$

The determinant of such a matrix $(a_\mu{}^\nu)$ is $+1$ and $a_0{}^0$ is positive, which indicates that we are restricted to proper orthochronous Lorentz transformations. Conversely, it is possible to find two unimodular matrices from equation (3.6) for a given proper orthochronous Lorentz transformation matrix $(a_\mu{}^\nu)$; they differ by an overall change of signs.

We also want to consider the antilinear transformations

$$\varphi'_{\mathrm{A}} = s_{\mathrm{A}}{}^{\dot{\mathrm{B}}}\,\varphi^*_{\dot{\mathrm{B}}} \tag{3.7}$$

we obtain the transformation matrix

$$a_\mu{}^\nu = \tfrac{1}{2} s^{*\mathrm{C}}_{\dot{\mathrm{A}}}\, s_{\mathrm{B}}{}^{\dot{\mathrm{D}}}\, \sigma^{\dot{\mathrm{A}}\mathrm{B}}_{\mu}\, \sigma^{\nu}_{\dot{\mathrm{D}}\mathrm{C}} \tag{3.8}$$

which corresponds to an improper orthochronous Lorentz transformation. Thus, it includes the parity transformation $\mathscr{P}$ on the coordinates; but the complex conjugation of the spinor field leads to the interchange of the positive- and negative-frequency parts. Consequently, we associate this type of transformation with $\mathscr{CP}$ where $\mathscr{C}$ stands for charge conjugation. We cannot obtain antichronous Lorentz transformations in this manner; in particular, multiplying the elements of s by i does not change the resulting $a_\mu{}^\nu$ in equations (3.6) and (3.8).‡

On the other hand, all homogeneous Lorentz transformations can be represented by linear transformations of bispinors. Under a Lorentz transformation, a spinor field $\psi(x)$ goes into

$$\psi'(x') = S\psi(x) \tag{3.9}$$

and the invariance of the γ_μ gives

$$a_\mu{}^\nu S\gamma_\nu S^{-1} = \gamma_\mu \tag{3.10}$$

whence

$$a_\mu{}^\nu = \tfrac{1}{4}\mathrm{Tr}(S^{-1}\gamma_\mu S\gamma^\nu) \tag{3.11}$$

The matrices that correspond to the parity transformation $\mathscr{P}$ and time reflection $\mathscr{T}'$ are

$$S_P = \pm\, i\gamma_0 \tag{3.12}$$

$$S_T = \pm\, i\gamma_1\gamma_2\gamma_3 \tag{3.13}$$

and they give the correct Lorentz transformation matrices through equation (3.11). Neither of these transformations involves complex conjugation of the field; thus, the second one represents the strong time reflection that interchanges particles and antiparticles when t changes into $-t$. In the representation (1.5) of the γ_μ, they are

$$S_P = \pm\, i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.14}$$

$$S_T = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.15}$$

and the transformations of the spinors are, choosing the plus sign,

$$\varphi' = i\chi \tag{3.16}$$

$$\chi' = i\varphi \tag{3.17}$$

for a parity transformation, and

$$\varphi' = \chi \tag{3.18}$$

$$\chi' = -\,\varphi \tag{3.19}$$

for time reflection. If we use equation (2.2), equation (3.16) becomes

$$\varphi'_{\mathrm{A}}(x') = \partial^{\dot{\mathrm{A}}\mathrm{B}}\varphi_{\mathrm{B}}(x)/m \tag{3.20}$$

‡ Aharoni (1965) relates the antichronous part of the Lorentz group to transformations with matrices of determinant −1; we believe this is incorrect.

it is debatable to what extent this relation represents the Lorentz transformation, especially if we consider that it involves not only the field but also its time derivative.‡

Charge conjugation, $\mathscr{C}$, which leaves the coordinates invariant, is represented by the antilinear transformation

$$\psi'(x) = \pm i\gamma_2 \psi^*(x) \tag{3.21}$$

so that $\mathscr{P}\mathscr{C}$ is represented by

$$\psi'(x') = \pm \gamma_0 \gamma_2 \psi^*(x) \tag{3.22}$$

whence

$$\varphi'(x') = \sigma_2 \varphi^*(x) \tag{3.23}$$

This transformation, up to a factor i, belongs to the type (3.7). We also note that what is usually called time reversal,

$$\psi'(x') = \pm \gamma_1 \gamma_3 \psi^*(x) \tag{3.24}$$

corresponds to $\mathscr{T} = \mathscr{T}'\mathscr{C}$ and implies no interchange of particles and antiparticles.

If we restrict ourselves to the vector space of spinor fields as functions of a three-vector variable, we have represented proper orthochronous Lorentz transformations and also the product of this subgroup with $\mathscr{P}\mathscr{C}$. Representation of $\mathscr{P}$ and $\mathscr{T}$ are in a sense tied to the dynamics of the field, as stated by equations (2.2) and (2.3). These properties of spinors make them particularly well suited to the formulation of weak interactions.

4. *Third-Order Equation*

The Dirac equation (2.1) can be obtained from the Lagrangian density

$$\mathscr{L}^{(1)} = \tfrac{1}{2}i(\bar{\psi}\gamma_\mu \psi_{,\mu} - \bar{\psi}_{,\mu}\gamma_\mu \psi) - m\bar{\psi}\psi \tag{4.1}$$

We eliminate χ by means of equation (2.2), and this Lagrangian density becomes

$$\begin{aligned}\mathscr{L} = \tfrac{1}{2}i[&(\varphi^*_{,\alpha}\sigma_\alpha \sigma_\mu \sigma_\beta \varphi_{,\beta\mu} - \varphi^*_{,\alpha\mu}\sigma_\alpha \sigma_\mu \sigma_\beta \varphi_{,\beta})/m^2 \\ &- \varphi^* \sigma_\alpha \varphi_{,\alpha} + \varphi^*_{,\alpha}\sigma_\alpha \varphi]\end{aligned} \tag{4.2}$$

It is not equivalent to $\mathscr{L}^{(1)}$, because we have used an equation of motion in the substitution; also the presence of second-order derivatives changes the nature of the restrictions on the variations of the fields at the boundary. The Euler–Lagrange equations, extended to this case (Barut & Mullen, 1962b; Goldberg & Marx, 1967), become

$$i\partial^{\dot{B}A}(\partial^2 + m^2)\varphi_A = 0 \tag{4.3}$$

the third-order equation considered by Kibble & Polkinghorne (1968) and

‡ Such a transformation is used by Theis (1959). He does not make any distinction between the spinor and bispinor equations in this context.

Barut & Mullen (1962a). These fields have then six basic degrees of freedom instead of the four for the Dirac field, and the equations are not equivalent.

We can separate the solutions of this equation into two parts,

$$\varphi_A = \xi_A + \eta_A \tag{4.4}$$

by means of projection operators; we set

$$\xi_A = -\,\partial^2 \varphi_A/m^2 \tag{4.5}$$

$$\eta_A = (\partial^2 + m^2)\,\varphi_A/m^2 \tag{4.6}$$

They satisfy

$$(\partial^2 + m^2)\,\xi_A = 0 \tag{4.7}$$

$$i\partial^{\dot{B}A}\eta_A = 0 \tag{4.8}$$

so that ξ corresponds to the field φ in equation (2.5) and η represents a massless spin-1/2 field.‡

We obtain the conserved densities from the Lagrangian density (4.2) via Noether's theorem.§ A guage transformation of the first kind gives the current density

$$j_\mu' = -\tfrac{1}{2}\varphi^*\sigma_\mu\varphi - \tfrac{1}{2}(\varphi^*\sigma_\mu\varphi_{,\alpha\alpha} + \varphi^*\sigma_\alpha\varphi_{,\alpha\mu} - \varphi^*_{,\mu}\sigma_\alpha\varphi_{,\alpha})/m^2 + \text{c.c.} \tag{4.9}$$

where c.c. stands for the complex conjugate of the expression preceding it. We express j_μ' in terms of ξ and η and separate it into two parts by adding the divergence of an antisymmetric tensor, $f_{\alpha\mu,\alpha}$, where

$$f_{\alpha\mu} = \tfrac{1}{4}\varphi^*(\sigma_\alpha\sigma_\mu - \sigma_\mu\sigma_\alpha)\,\sigma_\beta\,\varphi_{,\beta}/m^2 + \text{c.c.} \tag{4.10}$$

We obtain

$$j_\mu = \xi^*\sigma_\mu\xi + \xi^*_{,\alpha}\sigma_\alpha\sigma_\mu\sigma_\beta\,\xi_{,\beta}/m^2 - \eta^*\sigma_\mu\eta \tag{4.11}$$

Space-time translations yield the canonical stress-energy tensor

$$T_{\mu\nu}' = \tfrac{1}{2}i[\varphi^*_{,\nu}\sigma_\mu(\varphi + \varphi_{,\alpha\alpha}/m^2) + \varphi^*_{,\nu}\sigma_\alpha\varphi_{,\alpha\mu}/m^2 - \varphi^*_{,\mu\nu}\sigma_\alpha\varphi_{,\alpha}/m^2] + \text{c.c.} - \mathscr{L}g_{\mu\nu} \tag{4.12}$$

to which we add a term $f_{\alpha\mu\nu,\alpha}$, where

$$f_{\alpha\mu\nu} = \tfrac{1}{4}i\,[\varphi^*_{,\nu}(\sigma_\mu\sigma_\alpha - \sigma_\alpha\sigma_\mu)\,\sigma_\beta\,\varphi_{,\beta}/m^2 - 2\eta^*(g_{\mu\nu}\sigma_\alpha - g_{\alpha\nu}\sigma_\mu)\,\xi] + \text{c.c.} \tag{4.13}$$

to obtain

$$T_{\mu\nu} = \tfrac{1}{2}i[\xi^*\sigma_\mu\xi_{,\nu} + \xi^*_{,\alpha}\sigma_\alpha\sigma_\mu\sigma_\beta\,\xi_{,\beta\nu}/m^2 - \eta^*\sigma_\mu\eta_{,\nu}] + \text{c.c.} \tag{4.14}$$

‡ The parameters λ in Kibble & Polkinghorne (1958) and ε in Barut & Mullen (1962a) are simply normalization parameters for this massless field, as pointed out in the first reference. When they are set equal to zero, the effect is to eliminate this massless field again.

§ We use the expressions in Goldberg & Marx (1967), after changing the signs preceding $\partial\mathscr{L}/\partial\varphi_{k,\alpha\lambda}$ in equation (A.18) and $\partial_\lambda(A_{kl\rho\sigma}\varphi_l)$ in equation (A.20) from minus to plus.

Homogeneous Lorentz transformations lead to the angular momentum density

$$M'_{\mu\nu\rho} = x_\nu T'_{\mu\rho} - x_\rho T'_{\mu\nu} + S'_{\mu\nu\rho} \tag{4.15}$$

where

$$S'_{\mu\nu\rho} = \tfrac{1}{2}i\{\varphi^* \mathscr{S}^*_{\nu\rho}[\sigma_\mu(\varphi_{,\alpha\alpha}/m^2 + \varphi) + \sigma_\alpha \varphi_{,\alpha\mu}/m^2] - (g_{\mu\nu}\varphi^*_{,\nu} - g_{\mu\rho}\varphi^*_{,\nu} - \varphi^*_{,\mu}\mathscr{S}^*_{\nu\rho})\sigma_\beta \varphi_{,\beta}/m^2\} + \text{c.c.} \tag{4.16}$$

$$\mathscr{S}_{\mu\nu\text{A}}{}^{\text{B}} = \tfrac{1}{4}(\sigma_{\mu\dot{\text{C}}\text{A}}\sigma_\nu^{\dot{\text{C}}\text{B}} - \sigma_{\nu\dot{\text{C}}\text{A}}\sigma_\mu^{\dot{\text{C}}\text{B}}) \tag{4.17}$$

to which we add $f_{\alpha\mu\nu\rho,\alpha}$ where

$$\begin{aligned} f_{\alpha\mu\nu\rho} = x_\nu f_{\alpha\mu\rho} - x_\rho f_{\alpha\mu\nu} + \tfrac{1}{16}i\{&\varphi^*(\sigma_\alpha\sigma_\mu - \sigma_\mu\sigma_\alpha)(\sigma_\nu\sigma_\rho - \sigma_\rho\sigma_\nu)\sigma_\beta\varphi_{,\beta} \\ &- 4\varphi^*[\sigma_\alpha(g_{\mu\nu}\sigma_\rho - g_{\mu\rho}\sigma_\nu) - \sigma_\mu(g_{\alpha\nu}\sigma_\rho - g_{\alpha\rho}\sigma_\nu)]\sigma_\beta\varphi_{,\beta} \\ &- 2\varphi^*(\sigma_\alpha\sigma_\mu - \sigma_\mu\sigma_\alpha)(\sigma_\nu\varphi_{,\rho} - \sigma_\rho\varphi_{,\nu}) + 8\varphi^*(g_{\mu\nu}g_{\alpha\rho} - g_{\mu\rho}g_{\alpha\nu})\sigma_\beta\varphi_{,\beta} \\ &+ 4\varphi^*[(g_{\alpha\rho}\sigma_\nu - g_{\alpha\nu}\sigma_\rho)\varphi_{,\mu} - (g_{\mu\rho}\sigma_\nu - g_{\mu\nu}\sigma_\rho)\varphi_{,\alpha}] - \text{c.c.}\}/m^2 \end{aligned} \tag{4.18}$$

to obtain

$$M_{\mu\nu\rho} = x_\nu T_{\mu\rho} - x_\rho T_{\mu\nu} + S_{\mu\nu\rho} \tag{4.19}$$

where

$$\begin{aligned} S_{\mu\nu\rho} = \tfrac{1}{4}i[&\xi^*(\sigma_\mu\sigma_\nu\sigma_\rho - \sigma_\rho\sigma_\nu\sigma_\mu)\xi + \xi^*_{,\alpha}\sigma_\alpha(\sigma_\mu\sigma_\nu\sigma_\rho - \sigma_\rho\sigma_\nu\sigma_\mu)\sigma_\beta\xi_{,\beta}/m^2 \\ &- \eta^*(\sigma_\mu\sigma_\nu\sigma_\rho - \sigma_\rho\sigma_\nu\sigma_\mu)\eta] \end{aligned} \tag{4.20}$$

We note that the expressions for j_μ, $T_{\mu\nu}$ and $M_{\mu\nu\rho}$ are all the *difference* between two terms of precisely the form found in equations (2.8), (2.10) and (2.13), after equation (4.8) is used to eliminate some terms for the field η.

The solutions of equations (4.7) and (4.8) can be written in the form

$$\begin{aligned} \xi(x) = (2\pi)^{-3/2}\int d^3p \sum_\lambda \left[\frac{m^2}{2p_0(p_0 - \lambda|\mathbf{p}|)}\right]^{1/2} \\ \times [b_\lambda(\mathbf{p})\exp(-ip\cdot x) + d_\lambda(\mathbf{p})\exp(ip\cdot x)]\chi_\lambda(\hat{p}) \end{aligned} \tag{4.21}$$

$$\eta(x) = (2\pi)^{-3/2}\int d^3k\,[a(\mathbf{k})\exp(-ik\cdot x) + c(\mathbf{k})\exp(ik\cdot x)]\chi_+(\hat{k}) \tag{4.22}$$

where the χ_λ are the usual helical states and

$$p_0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{4.23}$$

$$k_0 = |\mathbf{k}| \tag{4.24}$$

We obtain j_0 from equation (4.11) and integrate it over all space; this gives the 'charge'

$$Q = \int d^3p(b^*_\lambda b_\lambda + d^*_\lambda d_\lambda) - \int d^3k(a^*a + c^*c) \tag{4.25}$$

Similarly, the integral of $T_{0\nu}$ gives the energy-momentum vector

$$P_\nu = \int d^3p\,p_\nu(b^*_\lambda b_\lambda - d^*_\lambda d_\lambda) - \int d^3k\,k_\nu(a^*a - c^*c) \tag{4.26}$$

The space–space components of the angular momentum tensor can be found and they separate into orbital and spin parts‡

$$L_{ij} = i\int d^3p\{p_j[\chi_\lambda^\dagger b_\lambda^* \,\partial(b_{\lambda'}\chi_{\lambda'})/\partial p_i + \chi_\lambda^\dagger d_\lambda^* \,\partial(d_{\lambda'}\chi_{\lambda'})/\partial p_i] - (i\leftrightarrow j)\} - i\int d^3k\{k_j \times [\chi_+^\dagger a^* \,\partial(a\chi_+)/\partial k_i + \chi_+^\dagger c^*\,\partial(c\chi_+)/\partial k_i] - (i\leftrightarrow j)\} \tag{4.27}$$

$$\mathbf{S} = \tfrac{1}{2}\int d^3p(\chi_\lambda^\dagger b_\lambda^* \boldsymbol{\sigma}\, b_{\lambda'}\chi_{\lambda'} + \chi_\lambda^\dagger d_\lambda^* \boldsymbol{\sigma}\, d_{\lambda'}\chi_{\lambda'}) - \tfrac{1}{2}\int d^3k(\chi_+^\dagger a^* \boldsymbol{\sigma}\, a\chi_+ + \chi_+^\dagger c^* \boldsymbol{\sigma}\, c\chi_+) \tag{4.28}$$

The sign of the contributions of the different fields is determined by the direction of propagation in time in relativistic quantum mechanics or by anticommutation of creation and annihilation operators in the usual quantum theory of fields.

If the field that is eliminated from the theory is φ, the Lagrangian density $\mathscr{L}$ in equation (4.2) has the same form, and equation (4.3) is replaced by

$$i\partial_{\dot{A}B}(\partial^2 + m^2)\chi^{\dot{A}} = 0 \tag{4.29}$$

The fields ξ and η have a superindex $\dot{A}$ instead of a subindex A, and equation (4.8) is replaced by

$$i\partial_{\dot{A}B}\eta^{\dot{A}} = 0 \tag{4.30}$$

Consequently, the expansion in equation (4.22) contains χ_- instead of χ_+, which changes the contribution to the spin in equation (4.28) accordingly.

5. *Electromagnetic Interactions*

We obtain the interaction of this spinor field with an electromagnetic field, given by the potential A_μ, through the substitution (2.6). From the Lagrangian density (4.2), we find

$$\mathscr{L} = \tfrac{1}{2}i\{[(D_\alpha^*\varphi^*)\sigma_\alpha\sigma_\mu\sigma_\beta D_\mu D_\beta\varphi - (D_\mu^* D_\alpha^*\varphi^*)\sigma_\alpha\sigma_\mu\sigma_\beta D_\beta\varphi]/m^2 - \varphi^*\sigma_\alpha D_\alpha\varphi + (D_\alpha^*\varphi^*)\sigma_\alpha\varphi\} \tag{5.1}$$

We note that the operators D_μ and D_ν do not commute, but

$$D_\mu D_\nu - D_\nu D_\mu = ie\,F_{\mu\nu} \tag{5.2}$$

and consequently

$$D_{\dot{C}A}D^{\dot{C}B} = \delta_A^B D^2 + ie\,F_{\mu\nu}\mathscr{S}_{\mu\nu A}{}^B \tag{5.3}$$

The equation of motion is

$$iD^{\dot{B}C}(D_{\dot{D}C}D^{\dot{D}A} + m^2\delta_C^A)\varphi_A = 0 \tag{5.4}$$

‡ We recall (Marx, 1968) that these parts do not correspond to the terms in equation (4.19), since $S_{\mu\nu\rho}$ is not a conserved density.

We can still separate the solutions of equation (5.4) into two parts given by

$$\xi_C = - D_{\dot{B}C} D^{\dot{B}A} \varphi_A / m^2 \tag{5.5}$$

$$\eta_C = (D_{\dot{B}C} D^{\dot{B}A} + m^2 \delta_C^A) \varphi_A / m^2 \tag{5.6}$$

which obey

$$(D_{\dot{B}C} D^{\dot{B}A} + m^2 \delta_C^A) \xi_A = 0 \tag{5.7}$$

$$i D^{\dot{B}A} \eta_A = 0 \tag{5.8}$$

Equation (5.3) shows that equation (5.7) can be written in the form

$$[(D^2 + m^2) \delta_B^A + ie F_{\mu\nu} \mathscr{S}_{\mu\nu B}{}^A] \xi_A = 0 \tag{5.9}$$

which is essentially Feynman's equation.

Equation (5.8) indicates that the massless field is charged, so that in this theory it should not be interpreted as the neutrino field.‡
We also find that the conserved current density is

$$j'_\mu = -\tfrac{1}{2}\varphi^* \sigma_\mu \varphi - \tfrac{1}{2}[\varphi^* \sigma_\mu \sigma_\alpha \sigma_\beta D_\alpha D_\beta \varphi + \varphi^* \sigma_\alpha D_\mu D_\alpha \varphi - (D_\mu^* \varphi^*) \sigma_\alpha D_\alpha \varphi]/m^2 + \text{c.c.} \tag{5.11}$$

which can be changed to

$$j_\mu = \xi^* \sigma_\mu \xi + (D_\alpha^* \xi^*) \sigma_\alpha \sigma_\mu \sigma_\beta D_\beta \xi / m^2 - \eta^* \sigma_\mu \eta \tag{5.12}$$

by adding the divergence of

$$f_{\alpha\mu} = \tfrac{1}{4}\varphi^* (\sigma_\alpha \sigma_\mu - \sigma_\mu \sigma_\alpha) \sigma_\beta D_\beta \varphi / m^2 + \text{c.c.} \tag{5.13}$$

The current density (5.12) separates into $j_\mu^{(1)}$ and $j_\mu^{(2)}$ given by

$$j_\mu = j_\mu^{(1)} - j_\mu^{(2)} \tag{5.14}$$

$$j_\mu^{(1)} = \xi^* \sigma_\mu \xi + (D_\alpha^* \xi^*) \sigma_\alpha \sigma_\mu \sigma_\beta D_\beta \xi / m^2 \tag{5.15}$$

and both parts are conserved independently. Furthermore, $j_\mu^{(1)}$ can be obtained from the current density (2.7) for a Dirac field that obeys

$$(-i\gamma \cdot D + m) \psi = 0 \tag{5.16}$$

The fact that the field η interacts with the electromagnetic field in this theory is to be expected if we consider that the basic field φ transforms in the usual way,

$$\varphi(x) \to \varphi'(x) = \varphi(x) \exp[-ie\Lambda(x)] \tag{5.17}$$

under gauge transformations of the second kind. It is possible to have ξ obey equation (5.7) and η the free-field equation (4.8), but then the unifying concept of the field φ is lost.

‡ It is not clear whether this difficulty is removed by the separation given by Barut & Mullen (1962a) in their footnote on page 194. It appears to lead to an equation different from (5.4), and the substitution

$$\partial_\mu \to D'_\mu = \partial_\mu + \tfrac{1}{2} ie(1 + \tau_3) A_\mu \tag{5.10}$$

is not Lorentz covariant.

A more desirable interpretation of Q in equation (4.25) is that of a lepton charge, which has to be conserved for electromagnetic and weak interactions.

In a dynamical problem, we can specify φ, $\dot{\varphi}$ and $\ddot{\varphi}$ initially (or ξ, $\dot{\xi}$ and η), and determine the fields at later times. In the context of relativistic quantum mechanics, it is appropriate to specify positive-frequency parts at the initial time and negative-frequency parts at the final time; this can be further modified by changing the Green function. The probability amplitudes in momentum space are given by generalizations of equations (4.21) and (4.22), such as

$$\xi(x) = (2\pi)^{-3/2} \int d^3p \sum_\lambda \left[\frac{m^2}{2p_0(p_0 - \lambda|\mathbf{p}|)}\right]^{1/2} \times [b_\lambda(\mathbf{p}, t) \exp(i\mathbf{p}\cdot\mathbf{x}) + d_\lambda(\mathbf{p}, t) \exp(-i\mathbf{p}\cdot\mathbf{x})] \chi_\lambda(\hat{p}) \qquad (5.18)$$

$$D_0 \xi(x) - \boldsymbol{\sigma}\cdot\mathbf{D}\xi(x) = i(2\pi)^{-3/2} \int d^3p \sum_\lambda \left[\frac{m^2(p_0 - \lambda|\mathbf{p}|)}{2p_0}\right]^{1/2} \times [-b_\lambda(\mathbf{p}, t) \exp(i\mathbf{p}\cdot\mathbf{x}) + d_\lambda(\mathbf{p}, t) \exp(-i\mathbf{p}\cdot\mathbf{x})] \chi_\lambda(\hat{p}) \qquad (5.19)$$

$$\eta(x) = (2\pi)^{-3/2} \int d^3k [a(\mathbf{k}, t) \exp(i\mathbf{k}\cdot\mathbf{x}) + c(\mathbf{k}, t) \exp(-i\mathbf{k}\cdot\mathbf{x})] \chi_+(\hat{k}) \qquad (5.20)$$

In terms of these time-dependent amplitudes, the charge Q in equation (4.25) has the same form.

We also note that the parity transformation given by equations (3.9) and (3.14) becomes dependent on the electromagnetic field, since they imply that

$$\varphi'_A(x') = D^{\dot{A}B} \varphi_B(x)/m \qquad (5.21)$$

a relation that again exhibits the dynamical nature of this symmetry.

6. *Weak Interactions*

We can use fields that obey a third-order equation to discuss weak interactions if we identify the massive part with the electron or muon and the massless part with the corresponding neutrino. Thus, we need two of these fields to describe a leptonic process such as muon decay.

The type of field that enters in such an interaction can be obtained from the usual four-fermion coupling. It involves currents of the form

$$J'_\mu = \bar{\psi}_e \gamma_\mu (1 + i\gamma_5)\psi_\nu + \text{c.c.} \qquad (6.1)$$

and, since our choice of γ-matrices implies that

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad (6.2)$$

the electron field enters through the part projected by the operator

$$\tfrac{1}{2}(1 + i\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.3}$$

This selects the field $\chi^{\dot{A}}$ from the bispinor. The current (6.1) is then a part of

$$J_\mu = \chi^{*A}\, \sigma_{\mu\dot{B}A}\, \chi^{\dot{B}} \tag{6.4}$$

which in addition contains terms with two electron fields or two neutrino fields. The nature of the muon field depends on our choice of the assignment of the μ^+ or μ^- to the role of particle. The analogy between the μ^- and the electron is usually taken as an indication that it should be the particle, but we prefer to assign the μ^+ to this role because it allows the neutrinos to be part of a single bispinor field, because it suggests a single lepton number assignment that forbids the decay of the muon into an electron plus one or more photons, and because the different mode of propagation could be involved in an explanation of the large mass ratio for the muon and electron.

If we use capital Greek letters for the muon field, we can write the Lagrangian density in the form

$$\mathscr{L} = \mathscr{L}_0(\chi) + \mathscr{L}_0(\Phi) + \mathscr{L}_I \tag{6.5}$$

where $\mathscr{L}_0$ is given by equation (4.2) and

$$\mathscr{L}_I = g\chi^{*A}\, \sigma_{\mu\dot{B}A}\, \chi^{\dot{B}}\, \Phi^*_{\dot{C}}\, \sigma_\mu^{\dot{C}D}\, \Phi_D = gJ_\mu(\chi) J_\mu(\Phi) \tag{6.6}$$

Since this interaction term contains no derivatives of the fields, the conserved electron and muon currents are still given by equation (4.9). The equations of motion have the form

$$i\partial_{\dot{B}A}(\partial^2 + m^2)\chi^{\dot{B}} = gm^2\, \sigma_{\mu\dot{B}A}\, \chi^{\dot{B}} J_\mu(\Phi) \tag{6.7}$$

and the electron and neutrino parts defined by equations (4.5) and (4.6) obey

$$(\partial^2 + m^2)\,\xi^{\dot{A}} = ig\partial^{\dot{A}B}[\chi^{\dot{C}} J_{\dot{C}B}(\Phi)] \tag{6.8}$$

$$i\partial_{\dot{A}B}\,\eta^{\dot{A}} = g\chi^{\dot{A}} J_{\dot{A}B}(\Phi) \tag{6.9}$$

Using these equations, we find that adding the divergence of $f_{\alpha\mu}$ from equation (4.10) gives a current density

$$\begin{aligned} j_\mu = {} & \xi^*\sigma_\mu\xi + \xi^*_{,\alpha}\sigma_\alpha\sigma_\mu\sigma_\beta\,\xi_{,\beta}/m^2 - \eta^*\sigma_\mu\eta \\ & + 2ig\{\partial_\alpha[J_\beta(\Phi)\chi^*]\,\sigma_\beta\sigma_\alpha\sigma_\mu\chi - \chi^*\sigma_\mu\sigma_\alpha\sigma_\beta\,\partial_\alpha[\chi J_\beta(\Phi)]\}/m^2 \\ & + ig[J_\alpha(\Phi)\chi^*\sigma_\alpha\sigma_\mu\sigma_\beta\,\xi_{,\beta} - \xi^*_{,\alpha}\sigma_\alpha\sigma_\mu\sigma_\beta\,\chi J_\beta(\Phi)]/m^2 \\ & + g^2 J_\alpha(\Phi) J_\beta(\Phi)\,\chi^*\sigma_\alpha\sigma_\mu\sigma_\beta\,\chi/m^2 \end{aligned} \tag{6.10}$$

which includes terms dependent on the interaction.

If we define the neutrino current density by

$$j^{(2)}_\mu = \eta^*\sigma_\mu\eta - H^*\sigma_\mu H, \tag{6.11}$$

we calculate

$$j^{(2)}_{\mu,\mu} = ig[J_\mu(\Phi)(\xi^*\sigma_\mu\eta - \eta^*\sigma_\mu\xi) - J_\mu(\chi)(\Xi^*\sigma_\mu H - H^*\sigma_\mu\Xi)] \tag{6.12}$$

and conclude that this current is not conserved. We are thus confronted with the difficulty to reconcile conservation of lepton number with conservation of charge for this form of the weak interactions.

7. *Relativistic Quantum Mechanics*

The interpretation of a classical field as a wave function in relativistic quantum mechanics is closely connected to the type of Green function used in the solution of a dynamical problem and the corresponding specification of boundary conditions.

The Green function for this third-order equation is a second-rank spinor that satisfies the equation

$$i\partial^{\dot{B}A}(\partial^2 + m^2)\, G_{\dot{C}A}(x, x') = -\,\delta(x - x')\,\delta^{\dot{B}}_{\dot{C}} \tag{7.1}$$

As in other cases, we can find the Fourier transform and invert it, that is, we determine

$$G_{\dot{C}A}(x, x') = \frac{1}{(2\pi)^4} \int\limits_{(C)} d^4 k \frac{k_\mu \sigma_{\mu\dot{C}A} \exp[-\,ik\cdot(x - x')]}{k^2(k^2 - m^2)} \tag{7.2}$$

where the path (C) specifies how we have to integrate around the poles of the integrand. The exact nature of the path to be selected depends on the identification of the positive- or negative-frequency part with the particles, which should be specified at the intial time. The usual choice is the causal Green function or Feynman propagator,

$$G^{(F)}_{\dot{C}A}(x, x') = \frac{1}{(2\pi)^4} \int d^4 k \frac{k_\mu \sigma_{\mu\dot{C}A} \exp[-ik\cdot(x - x')]}{(k^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} \tag{7.3}$$

which requires that the positive-frequency part of the wave function be specified at the initial time, and the negative-frequency part at the final time.

A process such as

$$\mu^- \to e^- + \bar{\nu}_e + \bar{\nu}_\mu \tag{7.4}$$

is determined by specifying the states of both antineutrinos at the final time and requiring that no positrons or μ^- be present at the final time and no neutrinos, electrons or μ^+ be present at the initial time. A perturbation expansion in powers of g can be used to determine the positive-frequency parts of both fields χ and Φ at the final time and the negative-frequency parts at the initial time. Those parts containing μ^- and e^- give the probability of the process under study and the amplitudes of the particles involved. Another process that would be described by the same calculation is

$$\mu^+ + \bar{\nu}_e \to \bar{\nu}_\mu + \nu_e \tag{7.5}$$

which violates charge conservation; such a reaction has to be forbidden by dynamical considerations or additional constraints.

As in the case of the Dirac equation, we have to change the signs of certain elements of the Hamiltonian (Marx, 1970c) in order to have a conserved charge for the massive particles. This can also be accomplished by a quantization of the field (Marx, 1972) that leads to a fixed number of 'particles'.

There are no special difficulties with purely electromagnetic interactions, since the currents for the massive and massless particles are separately conserved.

In the conventional approach to relativistic quantum mechanics (Feynman, 1949; Bjorken & Drell, 1964), problems with divergences should be helped by the degree of the integrand in equation (7.3), which is −3 rather than −1, as is the case for the usual fermion propagator.

8. *Quantization*

Much effort was spent (Kibble & Polkinghorne, 1958; Barut & Mullen, 1962a) in following a canonical quantization procedure for this field. This leads to difficulties such as 'improper limits', indefinite metric in the space of state vectors and subsidiary conditions.

Due to the intrinsic ambiguity in the passage from (antisymmetric) Poisson or Dirac brackets to (symmetric) anticommutators, we do not find a strong reason to rely on a canonical quantization procedure for a fermion field. We prefer the straightforward approach of identifying the independent amplitudes in momentum space, with normalization factors determined by the simple form of the charge, and assuming that they obey the usual anticommutation relations, such as

$$\{b_\lambda(\mathbf{p}), b_{\lambda'}^\dagger(\mathbf{p}')\} = \{d_\lambda(\mathbf{p}), d_{\lambda'}^\dagger(\mathbf{p}')\} = \delta_{\lambda\lambda'}\,\delta(\mathbf{p}-\mathbf{p}') \tag{8.1}$$

$$\{a(\mathbf{k}), a^\dagger(\mathbf{k}')\} = \{c(\mathbf{k}), c^\dagger(\mathbf{k}')\} = \delta(\mathbf{k}-\mathbf{k}') \tag{8.2}$$

while the others vanish. These are operators in the Schrödinger picture, and a detailed examination of possible problems with causality is needed in a particular theory.

In the conventional approach to quantum field theory, we assign the role of creation operator to the negative-frequency part of the field; operators are then normal-ordered, which changes the signs of certain terms in the conserved quantities. The state vector obeys the Schrödinger equation and we can use a retarded Green function to find the time development of the state.

In our modified quantization procedure (Marx, 1972), we develop a theory that is closely related to relativistic quantum mechanics. We assign the role of annihilation operators to both the positive- and negative-frequency parts of the field; in this manner, we only have to deal with a fixed number of 'particles'. We assume that the Hamiltonian operator displaces particles forward and antiparticles backward in time. This requires the introduction of a many-times formalism, and also changes the signs of the conserved quantities in the desired way. Such a theory makes use of a causal Green function in the space of state vectors. There also is a certain amount of flexibility that allows the interchange of the direction of propagation in time for some particles and their antiparticles, if this is required by the conservation laws. The main difficulty is presented by the consistency conditions in the case of interacting particles, since the equations of motion

$$\partial\psi/\partial t_1 = H_1\psi \tag{8.3}$$

$$\partial\psi/\partial t_2 = H_2\psi \tag{8.4}$$

imply that

$$[H_1, H_2]\psi = 0 \tag{8.5}$$

Barut & Mullen (1962a) present a detailed formulation of quantum electrodynamics for this field in such a way that the massless field is effectively eliminated, but they give no indication of its role in weak interactions.

9. *Concluding Remarks*

We have studied a classical two-component spinor field that obeys a third-order equation, starting from a Lagrangian density obtained from the Dirac theory for bispinors.

The free field can be decomposed into a massive field that obeys the Klein–Gordon equation and a massless field that obeys the Weyl equation. It is thus capable of describing the electron or muon field together with the corresponding neutrino. The electromagnetic interactions introduced through the usual gauge-invariant substitution maintain this separation, but the massless part of the field is also affected. This difficulty could be resolved by further research either to modify the interaction or to study the possibly peculiar properties of a massless charged field or particle (Bonnor, 1969).

The interaction Lagrangian density for weak interactions of leptons was obtained from the usual vector currents without derivatives of the fields, but in the present theory it includes terms with two massive fields or two massless fields, which introduce possible decays that violate electric charge conservation. Thus, a physically meaningful theory for weak *and* electromagnetic interactions still has to be found.

Fields that obey higher-order equations do not present any special difficulties of interpretation in relativistic quantum mechanics, but the signs of the contributions to conserved quantities may have to be changed by modifications similar to those used for the Dirac equation. It is also of interest to examine the changes in the usual perturbation theory; the improved convergence due to the degree –3 of the integrand of the propagator might solve some of the well-known problems with divergent diagrams.

The quantization of this field can be carried out in several different ways, and there seems to be no justification in spending a great deal of effort on canonical formulations. The approach most attractive to us relates directly to probability amplitudes and relativistic quantum mechanics. This avoids problems such as an indefinite metric in state vector space.

A further unification of the electron and muon fields into a bispinor field that obeys a third-order equation, briefly mentioned by Kibble & Polkinghorne (1958), is another interesting possibility. Once the electromagnetic and weak interactions are found, we might find out how different masses for electrons and muons arise starting from a single field with one mass parameter.

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